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Some remarks on the Clebsch's system

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Abstract

The almost Hamilton–Poisson realization, the stability problem, the existence of periodic solutions and the numerical integration via the Lie–Trotter integrator for the Clebsch system are discussed and some of their properties are pointed out.

MSC: 53D20

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1. Clebsch system and its almost Hamilton–Poisson realization

It is well known that the equations of motion of a rigid body in an ideal fluid are given by:

$$\begin{cases} \dot{x} = x \times \dfrac{\partial H}{\partial p}, \\ \dot{p} = x \times \dfrac{\partial H}{\partial x} + p \times \dfrac{\partial H}{\partial p}, \end{cases} \tag{1.1}$$

where $H \in C^\infty(\mathbb{R}^6, \mathbb{R})$ is a quadratic polynomial in x and p, [1–3].

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A nontrivial integrable case of Eq. (1.1) is the Clebsch case, where

$$H(x_1,x_2,x_3,p_1,p_2,p_3)=\frac{1}{2}(c_1x_1^2+c_2x_2^2+c_3x_3^2+b_1p_1^2+b_2p_2^2+b_3p_3^2)$$

and

$$\frac{c_2-c_3}{b_1}+\frac{c_3-c_1}{b_2}+\frac{c_1-c_2}{b_3}=0.$$

After an explicit change of variables the Clebsch system can be written in the following form

$$\begin{cases}\dot{x}=x\times p,\\ \dot{p}=x\times Ax,\end{cases}$$

where

$$\begin{aligned}&x=[x_1,x_2,x_3]^{\mathrm{t}},\\ &p=[p_1,p_2,p_3]^{\mathrm{t}},\\ &A=\operatorname{diag}[a_1,a_2,a_3],\\ &a_1,a_2,a_3\in\mathbb{R},\ a_1>0,\ a_2>0,\ a_3>0,\ a_1\neq a_2\neq a_3,\end{aligned}$$

[3], or equivalently:

$$\begin{cases}\dot{x}_1=x_2p_3-x_3p_2,\\ \dot{x}_2=x_3p_1-x_1p_3,\\ \dot{x}_3=x_1p_2-x_2p_1,\\ \dot{p}_1=(a_3-a_2)x_2x_3,\\ \dot{p}_2=(a_1-a_3)x_1x_3,\\ \dot{p}_3=(a_2-a_1)x_1x_2.\end{cases}\tag{1.2}$$

Remark 1.1. It is not hard to see that Eq. (1.2) can be obtained from Eq. (1.1) if we take:

$$H(x_1,x_2,x_3,p_1,p_2,p_3)=\frac{1}{2}(a_1x_1^2+a_2x_2^2+a_3x_3^2+p_1^2+p_2^2+p_3^2).$$

Let $\{\cdot\,,\cdot\}_{Cl}$ be the bracket operation on $C^\infty(\mathbb{R}^6,\mathbb{R})$ given by:

$$\{f,g\}_{Cl}\stackrel{\text{def}}{=}(\nabla f)^{\mathrm{t}}\Pi_{Cl}(\nabla g),\tag{1.3}$$

for each $f,g\in C^\infty(\mathbb{R}^6,\mathbb{R})$ and where

$$\Pi_{Cl}=\begin{bmatrix}0&0&0&0&-x_3&x_2\\0&0&0&x_3&0&-x_1\\0&0&0&-x_2&x_1&0\\0&-x_3&x_2&0&0&0\\x_3&0&-x_1&0&0&0\\-x_2&x_1&0&0&0&0\end{bmatrix}.$$

Then we have:

Proposition 1.1. *The bracket* (1.3) *is an almost Poisson structure on* $\mathbb{R}^6$.

Proof. Indeed, it is easy to see that the bracket (1.3) is bilinear, skew-symmetric and it satisfies the Leibniz rule. It does not satisfy in general the Jacobi identity. For instance, an easy computation shows us that:

$$\begin{aligned}&\{x_1, \{p_1, p_2\}_{Cl}\}_{Cl} + \{p_2, \{x_1, p_1\}_{Cl}\}_{Cl} + \{p_1, \{p_2, x_1\}_{Cl}\}_{Cl} \\ &\quad = \{p_1, x_3\}_{Cl} = x_2 \neq 0,\end{aligned}$$

as required. □

Proposition 1.2. *The smooth function* $C \in C^\infty(\mathbb{R}^6, \mathbb{R})$ *given by*:

$$C(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \tag{1.4}$$

is an almost-Casimir of our configuration $(\mathbb{R}^6, \{\cdot,\cdot\}_{Cl})$.

Proof. Indeed, an easy computation shows us that:

$$\{C, f\}_{Cl} = 0,$$

for each $f \in C^\infty(\mathbb{R}^6, \mathbb{R})$. □

Proposition 1.3. *The Clebsch system* (1.2) *has the following almost Hamilton–Poisson realization*:

$$\left(\mathbb{R}^6, \{\cdot,\cdot\}_{Cl}, H\right)$$

where $\{\cdot,\cdot\}_{Cl}$ *is given by relation* (1.3) *and* $H \in C^\infty(\mathbb{R}^6, \mathbb{R})$ *has the following expression*:

$$H(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}(a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + p_1^2 + p_2^2 + p_3^2). \tag{1.5}$$

Proof. Indeed, we have successively:

$$\begin{aligned}\Pi_{Cl} \cdot \nabla H &= \begin{bmatrix} 0 & 0 & 0 & 0 & -x_3 & x_2 \\ 0 & 0 & 0 & x_3 & 0 & -x_1 \\ 0 & 0 & 0 & -x_2 & x_1 & 0 \\ 0 & -x_3 & x_2 & 0 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 & 0 & 0 \\ -x_2 & x_1 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1x_1 \\ a_2x_2 \\ a_3x_3 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \\ &= \begin{bmatrix} x_2p_3 - x_3p_2 \\ x_3p_1 - x_1p_3 \\ x_1p_2 - x_2p_1 \\ (a_3-a_2)x_2x_3 \\ (a_1-a_3)x_1x_3 \\ (a_2-a_1)x_1x_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix}\end{aligned}$$

as required. □

An easy computation leads us also to:

Proposition 1.4. *The smooth functions $H, C \in C^\infty(\mathbb{R}^6, \mathbb{R})$ given respectively by* (1.5) *and* (1.4) *are constants of motion for the dynamics* (1.2).

2. Stability problem

It is not hard to see (or using eventually MAPLE V) that the equilibrium states of Clebsch's system are:

$$\begin{aligned}
&e_1^{MN} = (M, 0, 0, N, 0, 0), \quad M, N \in \mathbb{R},\\
&e_2^{MN} = (0, M, 0, 0, N, 0), \quad M, N \in \mathbb{R},\\
&e_3^{MN} = (0, 0, M, 0, 0, N), \quad M, N \in \mathbb{R},\\
&e_4^{0MNP} = (0, 0, 0, M, N, P), \quad M, N, P \in \mathbb{R}.
\end{aligned}$$

Then we have:

Proposition 2.1. *The equilibrium states e_1^{MN}, $M, N \in \mathbb{R}$, have the following behaviour*:

(i) *The equilibrium state e_1^{0N}, $N \in \mathbb{R}^*$, is always spectrally stable.*
(ii) *If $a_2 > a_1$ and $a_3 > a_1$, then the equilibrium state e_1^{M0}, $M \in \mathbb{R}^*$, is spectrally stable.*
(iii) *If $a_3 < a_1$ or $a_2 < a_1$, then the equilibrium state e_1^{M0}, $M \in \mathbb{R}^*$, is unstable.*

Proof. Let A be the matrix of the linear part of our system (1.2), i.e.

$$A = \left[\begin{array}{c|c|c|c|c|c}
0 & p_3 & -p_2 & 0 & -x_3 & x_2\\
-p_3 & 0 & p_1 & x_3 & 0 & -x_1\\
p_2 & -p_1 & 0 & -x_2 & x_1 & 0\\
0 & x_3(a_3-a_2) & x_2(a_3-a_2) & 0 & 0 & 0\\
x_3(a_1-a_3) & 0 & x_1(a_1-a_3) & 0 & 0 & 0\\
x_2(a_2-a_1) & x_1(a_2-a_1) & 0 & 0 & 0 & 0
\end{array}\right]$$

(i) It is easy to see that the characteristic [resp. minimal] polynomial of the matrix $A(e_1^{0N})$, $N \in \mathbb{R}^*$, is given by:

$$p_{A(e_1^{0N})}(x) = x^4(x^2 + N^2)$$

[resp.]

$$m_{A(e_1^{0N})}(x) = x(x^2 + N^2),$$

and then our assertion follows via the Lyapunov theorem [4].

(ii), (iii) An easy computation shows us that the characteristic [resp. minimal] polynomial of the matrix $A(e_1^{M0})$, $M \in \mathbb{R}^*$, is given by:

$$p_{A(e_1^{M0})}(x) = x^2\big[x^2 + M^2(a_3 - a_1)\big]\big[x^2 + M^2(a_2 - a_1)\big]$$

[resp.]

$$m_{A(e_1^{M0})}(x) = x\big[x^2 + M^2(a_3 - a_1)\big]\big[x^2 + M^2(a_2 - a_1)\big]$$

and the our assertions follow via the Lyapunov theorem [4]. □

Similar arguments leads us to:

Proposition 2.2. *The equilibrium states e_2^{MN}, $M, N \in \mathbb{R}$, have the following behaviour*:

(i) *The equilibrium state e_2^{0N}, $N \in \mathbb{R}^*$, is always spectrally stable.*
(ii) *If $a_3 > a_2$ and $a_1 > a_2$, then the equilibrium state e_2^{M0}, $M \in \mathbb{R}^*$, is spectrally stable.*
(iii) *If $a_3 < a_2$ or $a_1 < a_2$, then the equilibrium state e_2^{M0}, $M \in \mathbb{R}^*$, is unstable.* □

Proposition 2.3. *The equilibrium states e_3^{MN}, $M, N \in \mathbb{R}$, have the following behaviour*:

(i) *The equilibrium state e_3^{0N}, $N \in \mathbb{R}^*$, is always spectrally stable.*
(ii) *If $a_2 > a_3$ and $a_1 > a_3$, then the equilibrium state e_3^{M0}, $M \in \mathbb{R}^*$, is spectrally stable.*
(iii) *If $a_2 < a_3$ or $a_1 < a_3$, then the equilibrium state e_3^{M0}, $M \in \mathbb{R}^*$, is unstable.* □

Proposition 2.4. *The equilibrium states e_4^{0MNP}, $M, N, P \in \mathbb{R}$, is spectrally stable.*

Remark 2.1. The cases of the equilibrium states: $e_1^{MN}, e_2^{MN}, e_3^{MN}$, $M, N \in \mathbb{R}^*$ remain open. The computation are complicated and the results cannot be put in a simple form.

Now we shall begin to discuss the nonlinear stability problem.

Proposition 2.5. *The equilibrium state $e_0 = (0,0,0,0,0,0)$ is nonlinear stable.*

Proof. An easy computation shows us that the function $H \in C^\infty(\mathbb{R}^6, \mathbb{R})$ given by (1.5) is a Lyapunov function and then our assertion follows via the Lyapunov theorem [4]. □

Proposition 2.6. *The equilibrium state e_1^{M0}, $M \in \mathbb{R}^*$, is nonlinear stable if and only if $a_3 > a_1$ and $a_2 > a_1$.*

Proof. Let $H_\varphi \in C^\infty(\mathbb{R}^6, \mathbb{R})$ be the smooth function given by:

$$\begin{aligned} H_\varphi(x_1, x_2, x_3, p_1, p_2, p_3) &= \frac{1}{2}(a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + p_1^2 + p_2^2 + p_3^2) \\ &\quad + \varphi\left(\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\right), \end{aligned} \tag{2.1}$$

where $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then the first variation of H_φ is given by:

$$\begin{aligned} \delta H_\varphi &= a_1x_1\delta x_1 + a_2x_2\delta x_2 + a_3x_3\delta x_3 + p_1\delta p_1 + p_2\delta p_2 + p_3\delta p_3 \\ &\quad + \varphi'(x_1\delta x_1 + x_2\delta x_2 + x_3\delta x_3), \end{aligned}$$

where

$$\varphi' = \frac{\partial \varphi}{\partial(\frac{1}{2}(x_1^2 + x_2^2 + x_3^2))}.$$

Now, at least equilibrium interest we have:

$$\delta H_\varphi(M, 0, 0, 0, 0, 0) = 0,$$

if and only if:

$$\varphi'\left(\frac{1}{2}M^2\right) = -a_1. \tag{2.2}$$

Then, the second variation of H_φ is given by:

$$\begin{aligned}\delta^2 H_\varphi = {} & a_1(\delta x_1)^2 + a_2(\delta x_2)^2 + a_3(\delta x_3)^2 + (\delta p_1)^2 + (\delta p_2)^2 + (\delta p_3)^2 \\ & + \varphi'\big((\delta x_1)^2 + (\delta x_2)^2 + (\delta x_3)^2\big) \\ & + \varphi''(x_1\delta x_1 + x_2\delta x_2 + x_3\delta x_3)^2.\end{aligned}$$

At the equilibrium of interest we have via (2.2):

$$\begin{aligned}\delta^2 H_\varphi(M, 0, 0, 0, 0, 0) = {} & (a_2 - a_1)(\delta x_1)^2 + (a_3 - a_1)(\delta x_3)^2 \\ & + (\delta p_1)^2 + (\delta p_2)^2 + (\delta p_3)^2 \\ & + \varphi''\left(\frac{1}{2}M^2\right)M^2(\delta x_1)^2.\end{aligned}$$

It is positive definite if we can find $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that:

$$\varphi''\left(\frac{1}{2}M^2\right) > 0. \tag{2.3}$$

Such a φ is given for instance by:

$$\varphi(x) = \left(x - \frac{1}{2}M^2\right)^2 - a_1 x.$$

Now, it is easy to see that under the restrictions (2.2) and (2.3) the smooth function $L_\varphi \in C^\infty(\mathbb{R}^6, \mathbb{R})$ given by:

$$\begin{aligned}L_\varphi(x_1, x_2, x_3, p_1, p_2, p_3) = {} & H_\varphi(x_1, x_2, x_3, p_1, p_2, p_3) \\ & - \frac{1}{2}a_1 M^2 - \varphi\left(\frac{1}{2}M^2\right)\end{aligned} \tag{2.4}$$

is a Lyapunov function and then via the Lyapunov theorem [4], the equilibrium state e_1^{M0}, $M \in \mathbb{R}^*$, is nonlinear stable. □

Proposition 2.7. *The equilibrium state e_2^{M0}, $M \in \mathbb{R}^*$, is nonlinear stable if and only if $a_3 > a_2$ and $a_1 > a_2$.*

Proof. Let $H_\varphi \in C^\infty(\mathbb{R}^6, \mathbb{R})$ be the smooth function given by (2.1). Then at the equilibrium of interest we have:

$$\delta H_\varphi(0, M, 0, 0, 0, 0) = 0,$$

if and only if:

$$\varphi'\left(\frac{1}{2}M^2\right) = -a_2. \tag{2.5}$$

On the other hand, we have via (2.5):

$$\begin{aligned}\delta^2 H_\varphi(0, M, 0, 0, 0, 0) &= (a_1 - a_2)(\delta x_1)^2 + (a_3 - a_2)(\delta x_2)^2\\ &\quad + (\delta p_1)^2 + (\delta p_2)^2 + (\delta p_3)^2\\ &\quad + \varphi''\left(\frac{1}{2}M^2\right)M^2(\delta x_2)^2.\end{aligned}$$

It is positive definite if we can find $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that:

$$\varphi''\left(\frac{1}{2}M^2\right) > 0. \tag{2.6}$$

Such a φ is given for instance by:

$$\varphi(x) = \left(x - \frac{1}{2}M^2\right)^2 - a_2 x.$$

Now, it is easy to see that under the restrictions (2.5) and (2.6) the smooth function $K_\varphi \in C^\infty(\mathbb{R}^6, \mathbb{R})$ given by:

$$\begin{aligned}K_\varphi(x_1, x_2, x_3, p_1, p_2, p_3) &= H_\varphi(x_1, x_2, x_3, p_1, p_2, p_3)\\ &\quad - \frac{1}{2}a_2 M^2 - \varphi\left(\frac{1}{2}M^2\right)\end{aligned} \tag{2.7}$$

is a Lyapunov function and then via the Lyapunov theorem [4], the equilibrium state e_2^{M0}, $M \in \mathbb{R}^*$, is nonlinear stable. □

Proposition 2.8. *The equilibrium state e_3^{M0}, $M \in \mathbb{R}^*$, is nonlinear stable if and only if $a_2 > a_3$ and $a_1 > a_3$.*

Proof. Let $H_\varphi \in C^\infty(\mathbb{R}^6, \mathbb{R})$ be the smooth function given by (2.1). Then at the equilibrium of interest we have:

$$\delta H_\varphi(0, 0, M, 0, 0, 0) = 0,$$

if and only if:

$$\varphi'\left(\frac{1}{2}M^2\right) = -a_3. \tag{2.8}$$

On the other hand, we have via (2.8):

$$\begin{aligned}\delta^2 H_\varphi(0,0,M,0,0,0) &= (a_1-a_3)(\delta x_1)^2+(a_2-a_3)(\delta x_2)^2\\ &\quad+(\delta p_1)^2+(\delta p_2)^2+(\delta p_3)^2\\ &\quad+\varphi''\left(\frac{1}{2}M^2\right)M^2(\delta x_3)^2.\end{aligned}$$

It is positive definite if we can find $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ such that:

$$\varphi''\left(\frac{1}{2}M^2\right)>0. \tag{2.9}$$

Such a φ is given for instance by:

$$\varphi(x)=\left(x-\frac{1}{2}M^2\right)^2-a_3x.$$

Now, it is easy to see that under the restrictions (2.8) and (2.9) the smooth function $R_\varphi \in C^\infty(\mathbb{R}^6,\mathbb{R})$ given by:

$$\begin{aligned}R_\varphi(x_1,x_2,x_3,p_1,p_2,p_3) &= H_\varphi(x_1,x_2,x_3,p_1,p_2,p_3)\\ &\quad-\frac{1}{2}a_3M^2-\varphi\left(\frac{1}{2}M^2\right)\end{aligned} \tag{2.10}$$

is a Lyapunov function and then via the Lyapunov theorem [4], the equilibrium state e_3^{M0}, $M\in\mathbb{R}^*$, is nonlinear stable. □

Remark 2.2. It is an open problem to decide the nonlinear stability or unstability of the equilibrium states e_1^{MN}, e_2^{MN}, e_3^{MN}, $M,N\in\mathbb{R}^*$. In these cases the above method is inconclusive.

3. The existence of periodic solutions

Let $H\in C^\infty(\mathbb{R}^6,\mathbb{R})$ be the smooth function given by (1.5). Then we have:

(i) H is a constant of motion for the dynamics (1.2) (see Proposition 1.4).
(ii) $H(0,0,0,0,0,0)=0$.
(iii) $\delta H(0,0,0,0,0,0)=0$.
(iv) $\delta^2 H(0,0,0,0,0,0)=0$ is positive definite.

Then via the Moser theorem [5] we have:

Proposition 3.1. *For each ε sufficiently small, the integral surface*:

$$H(x_1,x_2,x_3,p_1,p_2,p_3)=\varepsilon^2$$

contains at least one periodic solution of the dynamics (1.2) *whose periods are closed to the periods of the corresponding linear system.*

Let $L\in C^\infty(\mathbb{R}^6,\mathbb{R})$ be the smooth function given by (2.4). Then we have:

(i) L_φ is a constant of motion for the dynamics (1.2) (see Proposition 1.4).
(ii) $H_\varphi(M,0,0,0,0,0)=0$.
(iii) $\delta L_\varphi(M,0,0,0,0,0)=0$.
(iv) Under the restrictions (2.2), (2.3) $\delta^2 L_\varphi(M,0,0,0,0,0)=0$ is positive definite.

Then via the Moser theorem [5] we have:

Proposition 3.2. *Under the restrictions* (2.2), (2.3)*, for each ε sufficiently small, the integral surface*:

$$L_\varphi(x_1,x_2,x_3,p_1,p_2,p_3)=\varepsilon^2$$

contains at least one periodic solution of the dynamics (1.2) *whose periods are closed to the periods of the corresponding linear system.*

Let $K\in C^\infty(\mathbb{R}^6,\mathbb{R})$ be the smooth function given by (2.7). Then we have:

(i) K_φ is a first integral of the dynamics (1.2) (see Proposition 1.4).
(ii) $K_\varphi(0,M,0,0,0,0)=0$.
(iii) $\delta K_\varphi(0,M,0,0,0,0)=0$.
(iv) Under the restrictions (2.5), (2.6) $\delta^2 K_\varphi(0,M,0,0,0,0)=0$ is positive definite.

Then via the Moser theorem [5] we have:

Proposition 3.3. *Under the restrictions* (2.5), (2.6)*, for each ε sufficiently small, the integral surface*:

$$K_\varphi(x_1,x_2,x_3,p_1,p_2,p_3)=\varepsilon^2$$

contains at least one periodic solution of the dynamics (1.2) *whose periods are closed to the periods of the corresponding linear system.*

Let $R\in C^\infty(\mathbb{R}^6,\mathbb{R})$ be the smooth function given by (2.10). Then we have:

(i) R_φ is a first integral of the dynamics (1.1) (see Proposition 1.4).
(ii) $R_\varphi(0,0,M,0,0,0)=0$.
(iii) $\delta R_\varphi(0,0,M,0,0,0)=0$.
(iv) Under the restrictions (2.8), (2.9) $\delta^2 L_\varphi(0,0,M,0,0,0)=0$ is positive definite.

Then via the Moser theorem [5] we have:

Proposition 3.4. *Under the restrictions* (2.8), (2.9)*, for each ε sufficiently small, the integral surface*:

$$R_\varphi(x_1,x_2,x_3,p_1,p_2,p_3)=\varepsilon^2$$

contains at least one periodic solution of the dynamics (1.2) *whose periods are closed to the periods of the corresponding linear system.*

4. Numerical integration via the Lie–Trotter integrator

For beginning let us observe that our Hamiltonian vector field splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3} + X_{H_4} + X_{H_5} + X_{H_6},$$

where

$$H_1(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}a_1x_1^2,$$

$$H_2(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}a_2x_2^2,$$

$$H_3(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}a_3x_3^2,$$

$$H_4(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}p_1^2,$$

$$H_5(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}p_2^2,$$

$$H_6(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}p_3^2.$$

Then the Lie–Trotter integrator can be written in the following form:

$$\begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \\ x_3^{n+1} \\ p_1^{n+1} \\ p_2^{n+1} \\ p_3^{n+1} \end{bmatrix} = A_1A_2A_3B_1B_2B_3 \begin{bmatrix} x_1^{n} \\ x_2^{n} \\ x_3^{n} \\ p_1^{n} \\ p_2^{n} \\ p_3^{n} \end{bmatrix}, \tag{4.1}$$

(see for details [6–8]), where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_1x_1(0)t & 0 & 1 & 0 \\ 0 & -a_1x_1(0)t & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a_2x_2(0)t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ a_2x_2(0)t & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_3 x_3(0)t & 0 & 1 & 0 & 0 \\ -a_3 x_3(0)t & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos p_1(0)t & \sin p_1(0)t & 0 & 0 & 0 \\ 0 & -\sin p_1(0)t & \cos p_1(0)t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \cos p_2(0)t & 0 & -\sin p_2(0)t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin p_2(0)t & 0 & \cos p_2(0)t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} \cos p_3(0)t & 0 & \sin p_3(0)t & 0 & 0 & 0 \\ -\sin p_3(0)t & \cos p_3(0)t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A long but straightforward computation, or using eventually MAPLE V leads us to:

Proposition 4.1. *The Lie–Trotter integrator* (4.1) *has the following properties*:

(i) *It does not preserve the almost Poisson structure Π_{Cl}.*
(ii) *If*:

$$x_1(0) = 0, \qquad x_2(0) = 0, \qquad x_3(0) = 0$$

then it preserve the almost Poisson structure Π_{Cl}.
(iii) *It is not energy-preserving.*

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